doi:10.1006/jmaa.2000.7083, available online at http://www.idealibrary.com on IDEAL®

# Strong Asymptotic Equivalence and Inversion of Functions in the Class $K_c$

Dragan Djurčić

*Technical Faculty, Svetog Save 65, 32000 Čačak, Yugoslavia*

Aleksandar Torgašev

*Faculty of Mathematics, Studentski trg 16a, 11000 Beograd, Yugoslavia*

*Submitted by H. M. Srivastava*

Received December 17, 1998

In this paper we consider the class of functions $K_c$ introduced by W. Matuszewska (1964, *Studia Math*. **24**, 271–279) and W. Matuszewska and W. Orlicz (1965, *Studia Math*. **26**, 11–24). As a main result we describe, in terms of the class $K_c$, when two strictly increasing functions, as well as their inverse functions, are asymptoticaly equivalent in the strong sense. This result also gives a proper characterization of the class $K_c$. 

## 1. INTRODUCTION AND RESULTS

The well-known regularly varying functions $f : [a, +\infty) \mapsto (0, +\infty)$ ($a > 0$) in the sense of Karamata are defined as measurable functions satisfying the condition

$$\lim_{x\to+\infty} \frac{f(\lambda x)}{f(x)} = r(\lambda) < +\infty, \qquad \lambda > 0. \tag{K}$$

Because of the analysis of the divergence processes, instead of the condition (K) a more general condition

$$\lim_{\substack{x\to+\infty \\ \lambda\to 1}} \frac{f(\lambda x)}{f(x)} = 1 \tag{Sch}$$

has been considered in many papers.



0022-247X/01 $35.00

It was done first by R. Schmidt in 1925 [14] in a sequential form. The last condition as a Tauberian condition played an important role in papers [8, 18, 19] by Stanojević and Grow, then in papers [15–17] by Stadtmüller and Trautner, and in many other papers related to the background and the applications of the theory of regularly varying functions (see, e.g., [5]). In particular, Berman [3, 4] called the continuous functions satisfying condition (Sch) "regularly oscillating" functions and found some important applications of such functions. Cline [6] also considered the condition (Sch) and developed the entire theory of functions satisfying this condition. He called such functions the "intermediate regularly varying" functions.

In papers [9–13], W. Matuszewska and W. Orlicz introduced and investigated a class of continuous functions on the interval $[0, +\infty)$, which they called the class of $\varphi$-functions.

DEFINITION 1. *A function $f$ is a $\varphi$-function if $f: [0, +\infty) \mapsto [0, +\infty)$, $f(0) = 0$, $f$ is continuous and nondecreasing on $[0, +\infty)$, and $f(x) \to +\infty$ as $x \to +\infty$.*

In [11, 13] they also introduced and investigated the class of functions $K_c$, which is a subclass of the class of all $\varphi$-functions.

DEFINITION 2. *$K_c$ is the set of all $\varphi$-functions $f$ with the property*

$$\lim_{x\to+\infty} \frac{f(\alpha(x)x)}{f(x)} = 1 \tag{1}$$

*for an arbitrary continuous function $\alpha: [0, +\infty) \mapsto (0, +\infty)$ tending to 1 as $x \to +\infty$.*

Denoting by

$$k_f(\lambda) = \overline{\lim_{x\to+\infty}} \frac{f(\lambda x)}{f(x)} \qquad (\lambda \in (0, +\infty))$$

the index function of an arbitrary $\varphi$-function $f$, we observe that from

$$\lim_{\substack{x\to\infty\\ \lambda\to1}} \frac{f(\lambda x)}{f(x)} = 1 \tag{1$'$}$$

it follows that $k_f(\lambda) < +\infty$ for every $\lambda > 0$. Hence we conclude that restriction of any $\varphi$-function $f$ on interval $[a, +\infty)$ $(a > 0)$, which satisfies the relation (1′), belongs to the class *ORV* of all *O*-regularly varying functions (see, e.g., [1, 5, 6]).

Using some results from the papers [6, 7], it is also not difficult to see that for an arbitrary $\varphi$-function $f$ the conditions (1) and (1′) are equivalent, and each of them is equivalent to the continuity of the corresponding index function $k_f(\lambda)$ $(\lambda > 0)$.

In the same papers $\mathcal{O}$-regularly varying functions with continuous index function were considered systematically.

Assume that $f$ and $g$ are two strictly increasing $\varphi$-functions. Since they are continuous and strictly increasing, the corresponding inverse functions $f^{-1}$ and $g^{-1}$ are also defined and they are also $\varphi$-functions. Further assume that

$$f(x) \sim g(x) \tag{2}$$

as $x \to +\infty$. We consider the question: Under which conditions do we also have that

$$f^{-1}(x) \sim g^{-1}(x) \tag{3}$$

as $x \to +\infty$?

The answer to this question is given by the following lemma. This lemma also gives a characterization of all strictly increasing functions from the class $K_c$.

LEMMA. (a) *Suppose that $f$ and $g$ are two strictly increasing $\varphi$-functions, at least one of the functions $f^{-1}$, $g^{-1}$ belongs to the class $K_c$, and relation* (2) *holds. Then also relation* (3) *holds.*

(b) *If $f$ is a strictly increasing $\varphi$-function and for every strictly increasing $\varphi$-function $g$ for which* (2) *is true, relation* (3) *also holds, then $f^{-1} \in K_c$ and for every such function $g$ the function $g^{-1} \in K_c$.*

Next, let $\mathcal{A}$ be the class of all nondecreasing functions $f:[a,+\infty) \mapsto (0,+\infty)$ $(a > 0)$, which are unbounded at $+\infty$. For any function $f \in \mathcal{A}$ the generalised inverse of $f$ is defined by

$$f^{\leftarrow}(x) = \inf\{y \geq a \mid f(y) > x\}, \qquad x \geq f(a). \tag{I}$$

We notice that some other forms of generalised inverses (see, e.g., [2, 5]) coincide to the inverse (I) on the set $\mathcal{A}$. In particular, for the continuous, strictly increasing functions from $\mathcal{A}$, we have that $f^{-1}(x) = f^{\leftarrow}(x)$ $(x \geq f(a))$.

Next, consider the relation of the strong asymptotic equivalence on the set $\mathcal{A}$. For any function $f \in \mathcal{A}$ let $[f]$ be the corresponding equivalence class of $f$. The following theorem says that for the inversion of functions defined by (I), the conclusions of the Lemma also hold if the considered functions are not necessary continuous and strictly increasing.

THEOREM. (a) *If $f, g \in \mathcal{A}$ and $[f] = [g]$ then $[f^{\leftarrow}] = [g^{\leftarrow}]$ whenever one of the functions $f^{\leftarrow}$, $g^{\leftarrow}$ satisfies the condition* (Sch).

(b) *If $f \in \mathcal{A}$ and if $[f^{\leftarrow}] = [g^{\leftarrow}]$ whenever $g \in \mathcal{A}$ and $[f] = [g]$ then $f^{\leftarrow}$ and $g^{\leftarrow}$ for any such $g$ satisfy the condition* (Sch).

Part (a) of this Theorem as well as part (a) of the Lemma is motivated by the corresponding result by Balkema et al. [2]. In order to generalize this result (see also [5, p. 190, 14.(ii) and (iii)]), we involve the next known class of functions.

DEFINITION 3. *$K_c^*$ is the class of all functions $f \in K_c$ which for all $\lambda > 1$ and all $x \geq x_0(\lambda)$ satisfy the condition $f(x) \cdot c(\lambda) \leq f(\lambda x)$ for a function $c(\lambda)$ $(\lambda > 1)$ such that $c(\lambda) > 1$.*

It is easily seen that the class $K_c^*$ does not contain the functions from the class $K_c$ for which

$$\varliminf_{x \to +\infty} \frac{f(\lambda x)}{f(x)} = 1$$

for all $\lambda \in [1, \delta]$, where $\delta$ is a fixed number greater than 1, i.e., the functions $f$ for which $k_f(\lambda) = 1$ for all $\lambda \in [1/\delta, 1]$.

Any $\varphi$-function whose restriction on the interval $[a, +\infty)$ $(a > 0)$ is in the class $SRV$ belongs to the class $K_c$, but does not belong to the class $K_c^*$. In particular, the class $K_c^*$ contains all $\varphi$-functions whose restriction to the interval $[a, +\infty)$ $(a > 0)$ belongs to the class $RV$ and whose index $\rho > 0$.

By some results from [11, pp. 276–277] we get that for every strictly increasing function $f \in K_c^*$ there holds $f^{-1} \in K_c^*$, i.e., that class of all strictly increasing functions in $K_c^*$ is invariant by inverses, and for any strictly increasing function $f \in K_c \setminus K_c^*$ it holds $f^{-1} \notin K_c$.

Consequently, making use of the Lemma we immediately get the next corollary.

COROLLARY. *If $f$ and $g$ are strictly increasing $\varphi$-functions and $g \in K_c^*$, then $f(x) \sim g(x)$ as $x \to +\infty$ if and only if $f^{-1}(x) \sim g^{-1}(x)$ as $x \to +\infty$.*

It is easy to see that if $f$ and $g$ are strictly increasing $\varphi$-functions, then the statement of the Corollary holds if and only if $g \in K_c^*$.

The above Corollary offers in a sense maximal generalization of the corresponding result of Balkema et al. [2] (see [5, p. 190]).

## 2. PROOFS OF THE RESULTS

*Proof of the Lemma.* (a) Without loss of generality we can assume that $g^{-1} \in K_c$. Assuming next that $f(x) \sim g(x)$ as $x \to +\infty$ we have

$$\varliminf_{x \to +\infty} \frac{f(x)}{g(x)} = 1.$$

Whence for any fixed $\lambda > 1$ and all sufficiently large $x$ we find

$$f(x) \geq \frac{1}{\lambda} g(x).$$

Now for $x$ such that $f^{-1}(x)$ is as large as is needed we have $f^{-1}(x) \leq g^{-1}(\lambda x)$; whence

$$\frac{f^{-1}(x)}{g^{-1}(x)} \leq \frac{g^{-1}(\lambda x)}{g^{-1}(x)}.$$

Hence we obtain that

$$\overline{\lim_{x\to+\infty}} \frac{f^{-1}(x)}{g^{-1}(x)} \leq \overline{\lim_{x\to+\infty}} \frac{g^{-1}(\lambda x)}{g^{-1}(x)} = k_{g^{-1}}(\lambda).$$

Since by assumption $g^{-1} \in K_c$, we have that there holds $k_{g^{-1}}(\lambda) < +\infty$ for every $\lambda > 0$ and $\lim_{\lambda\to 1+} k_{g^{-1}}(\lambda) = 1$. Hence

$$\overline{\lim_{x\to+\infty}} \frac{f^{-1}(x)}{g^{-1}(x)} \leq 1.$$

Analogously we can find that

$$\underline{\lim_{x\to+\infty}} \frac{f^{-1}(x)}{g^{-1}(x)} \geq 1,$$

and consequently

$$\lim_{x\to+\infty} \frac{f^{-1}(x)}{g^{-1}(x)} = 1.$$

So (3) holds.

(b) Now assume that $f$ and $g$ satisfy assumptions from (b). By relation (2) we have that

$$1 = \lim_{x\to+\infty} \frac{f^{-1}(x)}{g^{-1}(x)} = \lim_{t\to+\infty} \frac{f^{-1}(g(t))}{g^{-1}(g(t))}$$
$$= \lim_{t\to+\infty} \frac{f^{-1}(g(t))}{t} = \lim_{t\to+\infty} \frac{f^{-1}\big((g(t)/f(t))f(t)\big)}{f^{-1}(f(t))}.$$

Next let $\alpha(t)$ be an arbitrary positive continuous function defined on interval $[0, +\infty)$ such that $\alpha(t) \geq 1$ $(t \geq 0)$ and $\alpha(t) \to 1+$ as $t \to +\infty$. Consider the function $\beta(t) = \alpha(f(t))$ $(t \geq 0)$.

If the function $h(t) = f(t)\beta(t)$ $(t \geq 0)$ is strictly increasing, then it is a $\varphi$-function. Since then $f(t) \sim h(t)$ as $t \to +\infty$, we find that

$$1 = \lim_{t\to+\infty} \frac{f^{-1}(\beta(t)f(t))}{f^{-1}(f(t))} = \lim_{t\to+\infty} \frac{f^{-1}(\alpha(f(t))f(t))}{f^{-1}(f(t))} = \lim_{p\to+\infty} \frac{f^{-1}(\alpha(p)\cdot p)}{f^{-1}(p)}.$$

If $h(t)$ $(t \geq 0)$ is not strictly increasing, we shall consider the function

$$r(t) = \max_{0 \leq x \leq t} h(x), \quad t \geq 0.$$

It is continuous and nondecreasing for all $t \geq 0$ and satisfies $r(0) = 0$, $r(t) \to +\infty$ as $t \to +\infty$, and $r(t) \geq \beta(t) \cdot f(t)$ for every $t \geq 0$. Besides, we shall prove that $r(t) \sim f(t)$ as $t \to +\infty$.

Let $\varepsilon > 0$. Then there is a $t_0 \geq 0$ such that

$$1 \leq h(t)/f(t) < 1 + \varepsilon,$$

and also there is a $t_1 > t_0$ such that

$$h(t) \geq \max_{0 \leq u \leq t_0} h(u)$$

for all $t \geq t_1$. Then for every $t \geq t_1$ and a function $v(t) \in [t_0, t]$ we have

$$\begin{aligned} 1 \leq \frac{r(t)}{f(t)} &= \frac{1}{f(t)} \max_{0 \leq u \leq t} h(u) = \frac{1}{f(t)} \max_{t_0 \leq u \leq t} h(u) \\ &= \frac{h(v(t))}{f(t)} \leq \frac{h(v(t))}{f(v(t))} < 1 + \varepsilon. \end{aligned}$$

Hence we find that $r(t) \sim f(t)$ as $t \to +\infty$. Next we define a function $r_1(t)$ $(t \geq 0)$ by $r_1(t) = r(t) + u(t)$, where $u(t) = t/4$ if $0 \leq t \leq 2$ and $u(t) = 1 - 1/t$ if $t \geq 2$. Then $r_1(t)$ is a strictly increasing $\varphi$-function which satisfies $r_1(t) \sim r(t) \sim f(t)$ as $t \to +\infty$. So we get

$$\begin{aligned} 1 &\leq \varliminf_{t \to +\infty} \frac{f^{-1}(\beta(t)f(t))}{f^{-1}(f(t))} \leq \varlimsup_{t \to +\infty} \frac{f^{-1}(\beta(t)f(t))}{f^{-1}(f(t))} \\ &\leq \lim_{t \to +\infty} \frac{f^{-1}\left(\frac{r_1(t)}{f(t)} f(t)\right)}{f^{-1}(f(t))} = 1. \end{aligned}$$

Consequently

$$\lim_{t \to +\infty} \frac{f^{-1}(\beta(t)f(t))}{f^{-1}(f(t))} = 1.$$

Hence, by using a similar procedure as before, we conclude that

$$\lim_{p \to +\infty} \frac{f^{-1}(\alpha(p) \cdot p)}{f^{-1}(p)} = 1,$$

which means that $f^{-1} \in K_c$.

Finally, since $f^{-1}(x) \sim g^{-1}(x)$ as $x \to +\infty$, we conclude that also $g^{-1} \in K_c$. ∎

*Proof of the Theorem.* (a) This proof is exactly the same as Lemma (a).

(b) Taking any function $f \in \mathscr{A}$, consider the functions

$$g_1(x) = f(x) - \frac{1}{x}, \quad g_2(x) = f(x) + 1 - \frac{1}{x} \quad (x \geq a).$$

Then $g_1(x), g_2(x) \in \mathscr{A}$, both functions are strictly increasing and we have

$$g_1(x) \leq f(x) \leq g_2(x) \quad (x \geq a), \quad g_1(x) \sim f(x) \sim g_2(x) \quad (\text{as } x \to +\infty).$$

Consequently $g_1^{\leftarrow}(x) \sim g_2^{\leftarrow}(x)$ as $x \to +\infty$. Hence, using the same proof as in Lemma (b) we find that both $g_1^{\leftarrow}$, $g_2^{\leftarrow}$ satisfy the condition (Sch). Since

$$g_2^{\leftarrow}(x) \leq f^{\leftarrow}(x) \leq g_1^{\leftarrow}(x) \qquad (x \geq a),$$

we find that $f^{\leftarrow}(x) \sim g_1(x)$ as $x \to +\infty$. Hence $f^{\leftarrow}$ satisfies the condition (Sch).

Finally, using very similar procedures, we find that for any function $g \in \mathscr{A}$ such that $[f] = [g]$ and $[f^{\leftarrow}] = [g^{\leftarrow}]$, the function $g^{\leftarrow}$ satisfies condition (Sch). ∎

## REFERENCES


1. S. Aljančić and D. Arandjelović, $\mathscr{O}$-regularly varying functions, *Publ. Inst. Math. (Beograd)* **22**, No. 36 (1977), 5–22.
2. A. A. Balkema, J. L. Geluk, and L. de Haan, An extension of Karamata's Tauberian theorem and its connection with complementary convex functions, *Quart. J. Math. Oxford Ser. (2)* **30** (1979), 385–416.
3. S. Berman, Sojourns and extremes of a diffusion process on a fixed interval, *Adv. in Appl. Probab.* **14** (1982), 811–832.
4. S. Berman, The tail of the convolution of densities and its application to a model of HIV-latency time, *Ann. Appl. Probab.* **2** (1992), 481–502.
5. N. H. Bingham, C. M. Goldie, and J. L. Teugels, "Regular Variation," Cambridge Univ. Press, Cambridge, UK, 1987.
6. D. B. H. Cline, Intermediate regular and Π-variation, *Proc. London Math. Soc. (3)* **68** (1994), 594–616.
7. D. Djurčić, $\mathscr{O}$-regularly varying functions and strong asymptotic equivalence, *J. Math. Anal. Appl.* **220** (1998), 451–461.
8. D. Grow and Č. V. Stanojević, Convergence and the Fourier character of trigonometric transforms with slowly varying convergence moduli, *Math. Ann.* **302** (1995), 433–472.
9. W. Matuszewska, Some further properties of $\varphi$-functions, *Bull. Acad. Pol. Sci.* **9** (1961), 445–450.
10. W. Matuszewska, Regularly increasing functions in connection with the theory of $L^{*\varphi}$-spaces, *Studia Math.* **21** (1962), 317–344.
11. W. Matuszewska, On a generalization of regularly increasing functions, *Studia Math.* **24** (1964), 271–279.
12. W. Matuszewska and W. Orlicz, On certain properties of $\varphi$-functions, *Bull. Acad. Pol. Sci.* **8** (1960), 439–443.

13. W. Matuszewska and W. Orlicz, On some classes of functions with regard to their orders of growth, *Studia Math.* **26** (1965), 11–24.
14. R. Schmidt, Über divergente Folgen und lineare Mittelbildungen, *Math. Z.* **22** (1925), 89–152.
15. U. Stadtmüller and R. Trautner, Tauberian theorems for Laplace tranforms, *J. Reine Angew. Math.* **310** (1979), 283–290.
16. U. Stadtmüller and R. Trautner, Tauberian theorems for Laplace transforms in dimension $D > 1$, *J. Reine Angew. Math.* **323** (1981), 127–138.
17. U. Stadtmüller and R. Trautner, Ratio Tauberian theorems for Laplace transforms without monotonicity assumptions, *Quart. J. Math. Oxford Ser. (2)* **36** (1985), 363–381.
18. Č. V. Stanojević, $\mathscr{O}$-regularly varying convergence moduli of Fourier and Fourier–Stieltjes series, *Math. Ann.* **279** (1987), 103–115.
19. Č. V. Stanojević, Structure of Fourier and Fourier–Stieltjes coefficients of series with slowly varying convergence moduli, *Bull. Amer. Math. Soc.* **19** (1988), 283–286.